

One-Dimensional Geometric Random Graphs With Nonvanishing Densities—Part I: A Strong Zero-One Law for Connectivity

Guang Han and Armand M. Makowski, *Fellow, IEEE*

Abstract—We consider a collection of n independent points which are distributed on the unit interval $[0, 1]$ according to some probability distribution function F . Two nodes are said to be adjacent if their distance is less than some given threshold value. When F admits a nonvanishing density f , we show under a weak continuity assumption on f that the property of graph connectivity for the induced geometric random graph exhibits a strong zero-one law, and we identify the corresponding critical scaling. This is achieved by generalizing to nonuniform distributions a limit result obtained by Lévy for maximal spacings under the uniform distribution.

Index Terms—Connectivity, critical scalings, geometric random graphs, nonuniform node placement, nonvanishing densities, zero-one laws.

I. INTRODUCTION

STARTING with a recent paper by Gupta and Kumar [11], there has been renewed interest in geometric random graphs [21] as models for wireless networks. Although much of the subsequent work has been carried out in dimension two (and higher), the one-dimensional case has also received some attention, e.g., see [4], [6]–[10], [12], [13], [15], [18], [19], [24], [25] (and references therein).

Most of these references deal with the following situation. The network comprises n nodes which are distributed independently and *uniformly* on the interval $[0, 1]$. Two nodes are then said to communicate with each other if their distance is less than some transmission range $\rho > 0$. In this setting the property of network connectivity (for the induced geometric random graph) is known to admit strong zero-one laws with a sharp phase transition [4], [6], [7], [9], [12], [13], [15], [19].

In this paper, we consider the case when the nodes are placed independently on the interval $[0, 1]$ according to some probability distribution F . We only assume that F admits a nonvan-

ishing density f with a weak continuity condition. Under these assumptions we show that the property of network connectivity also obeys a strong zero-one law, given in Theorem 3.1, and we identify the corresponding critical threshold. This answers an open problem stated in [19].

We approach this problem through the asymptotic properties of the maximal spacings under F . The main technical contribution of the paper is contained in Proposition 4.1, and constitutes an extension to nonuniform distributions of a well-known asymptotic result for maximal spacings obtained by Lévy under the uniform distribution [5], [17]. The limiting result obtained here is related to earlier results of Deheuvels [3, Theorem 4, p. 1183], and can be viewed as a one-dimensional version of a strong law derived by Penrose in dimension two (and higher) [20].

The paper is organized as follows: The network model and the assumptions on F are presented in Section II. The main result, Theorem 3.1, is discussed in Section III, and in Section IV we show its equivalence with Proposition 4.1. The proof of Proposition 4.1 is then developed in the next three sections: In Section V we relate the maximal spacings under F to the order statistics induced by independent uniformly distributed random variables (rvs). In Section VI we recall how these order statistics associated with the uniform distribution can be represented in terms of independent and identically distributed (i.i.d.) exponentially distributed rvs. This representation is a key ingredient of the proof of Proposition 4.1 given in Section VII. We conclude in Section VIII with various remarks concerning the results discussed in this paper.

A word on notation and conventions: All limiting statements, including asymptotic equivalences, are understood with n going to infinity. Almost everywhere is abbreviated as a.e. and all such statements are understood with respect to Lebesgue measure λ on the unit interval $[0, 1]$. The rvs under consideration are all defined on the same probability triple $(\Omega, \mathcal{F}, \mathbb{P})$. Probabilistic statements are made with respect to the probability measure \mathbb{P} , and we denote the corresponding expectation operator by \mathbb{E} . The notation $\xrightarrow{\mathbb{P}}_n$ (respectively, \Rightarrow_n) is used to signify convergence in probability (respectively, convergence in distribution) with n going to infinity. Also, we use the notation $=_{st}$ to indicate distributional equality.

II. MODEL AND ASSUMPTIONS

Throughout, let $\{X_i, i = 1, 2, \dots\}$ denote a sequence of i.i.d. rvs which are distributed on the unit interval $[0, 1]$ according to some common probability distribution function F . For each $n =$

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G. Han was with the Department of Electrical and Computer Engineering, and the Institute for Systems Research, University of Maryland, College Park, MD 20742 USA. He is now with Networks Advanced Technologies, Motorola, Inc., Arlington Heights, IL 60004 USA (e-mail: guang.han@motorola.com).

A. M. Makowski is with the Department of Electrical and Computer Engineering, and the Institute for Systems Research, University of Maryland, College Park, MD 20742 USA (e-mail: armand@isr.umd.edu).

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$2, 3, \dots$, we think of X_1, \dots, X_n as the locations of n nodes, labelled $1, \dots, n$, in the interval $[0, 1]$. Given a fixed distance or transmission range $\rho > 0$, two nodes are said to be adjacent if their distance is at most ρ , i.e., nodes i and j are adjacent if $|X_i - X_j| \leq \rho$, in which case an undirected edge is said to exist between them. The relevance of this model to wireless networking is discussed in Section VIII-A.

This notion of adjacency gives rise to an undirected geometric random graph on the set of nodes $\{1, \dots, n\}$, thereafter denoted by $\mathbb{G}(n; \rho)$. As usual, $\mathbb{G}(n; \rho)$ is said to be connected if every pair of nodes can be linked by at least one path over the edges of the graph. The probability of graph connectivity is simply

$$P(n; \rho) := \mathbb{P}[\mathbb{G}(n; \rho) \text{ is connected}].$$

Obviously $P(n; \rho) = 1$ whenever $\rho \geq 1$.

A number of assumptions are imposed on F . The most basic one is given first.

Assumption 1: The distribution $F : [0, 1] \rightarrow [0, 1]$ is absolutely continuous (with respect to λ).

Thus, F is differentiable a.e. on $[0, 1]$ with $F(0) = 0$ and $F(1) = 1$, and the relation

$$F(x) = \int_0^x f(t)dt, \quad x \in [0, 1] \quad (1)$$

holds for some density function $f : [0, 1] \rightarrow \mathbb{R}_+$. This density f is determined up to a.e. equivalence [27, Sec. 9.2].

The essential infimum¹

$$f_* := \text{ess inf}(f(x), x \in [0, 1])$$

is uniquely determined by F , hence by (the equivalence class of) f . There is no loss of generality in selecting (as we do from now on) the density f which appears in (1) so that

$$f_* = \inf(f(x), x \in [0, 1]). \quad (2)$$

This can be achieved by suitably redefining f on a set of zero Lebesgue measure, and will not affect the results obtained here since this procedure leaves F unchanged.

It is plain that $0 \leq f_* \leq 1$ with $f_* = 1$ corresponding to the case when F is the uniform distribution. Our main assumption requires the density f to be bounded away from zero in the following technical sense.

Assumption 2: With the density f selected such that (2) holds, there exists x_* in the interval $[0, 1]$ such that

$$f_* = f(x_*) > 0, \quad (3)$$

and this point x_* is a point of continuity for f .

The condition $f_* > 0$ amounts to the distribution function F being strictly increasing. Note that the minimizer appearing in Assumption 2 is not necessarily unique. However, the continuity

¹Recall that

$$f_* = \sup(a \in \mathbb{R} : \lambda(\{x \in [0, 1] : f(x) < a\}) = 0).$$

exhibited by f at such a minimizer x_* implies that for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|f(x) - f(x_*)| \leq \varepsilon \quad (4)$$

whenever $|x - x_*| \leq \delta$ in $[0, 1]$.

III. THE MAIN RESULT

A scaling is any mapping $\rho : \mathbb{N}_0 \rightarrow \mathbb{R}_+$. Of particular interest is the scaling $\rho^* : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ defined by

$$\rho_n^* = \frac{\log n}{n}, \quad n = 1, 2, \dots \quad (5)$$

As the next result shows, this scaling occupies a special place in the context of zero-one laws for graph connectivity in $\mathbb{G}(n; \rho)$.

Theorem 3.1: Assumptions 1 and 2 are enforced on F . For any scaling $\rho : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ such that

$$\lim_{n \rightarrow \infty} \frac{\rho_n}{\rho_n^*} = \frac{c}{f_*} \quad (6)$$

for some $c > 0$, we have

$$\lim_{n \rightarrow \infty} P(n; \rho_n) = \begin{cases} 0 & \text{if } 0 < c < 1 \\ 1 & \text{if } 1 < c. \end{cases} \quad (7)$$

This zero-one law is sometimes given in the following seemingly weaker, but equivalent, form.

Corollary 3.2: Under the assumptions of Theorem 3.1, the convergence (7) under (6) is equivalent to

$$\lim_{n \rightarrow \infty} P\left(n; \frac{c}{f_*} \rho_n^*\right) = \begin{cases} 0 & \text{if } 0 < c < 1 \\ 1 & \text{if } 1 < c. \end{cases} \quad (8)$$

Proof: We need only show that the zero-one law (8) implies the convergence (7) under (6). Thus, pick a scaling $\rho : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ such that (6) holds for some $c > 0$. In that case, for every ε in $(0, 1)$, there exists a positive integer $n^*(\varepsilon)$ such that

$$\frac{(1-\varepsilon)c}{f_*} \rho_n^* \leq \rho_n \leq \frac{(1+\varepsilon)c}{f_*} \rho_n^*$$

for all $n \geq n^*(\varepsilon)$. For each $n = 2, 3, \dots$, the function $\rho \rightarrow P(n; \rho)$ is monotone increasing on $[0, 1]$ so that

$$P\left(n; \frac{(1-\varepsilon)c}{f_*} \rho_n^*\right) \leq P(n; \rho_n) \quad (9)$$

and

$$P(n; \rho_n) \leq P\left(n; \frac{(1+\varepsilon)c}{f_*} \rho_n^*\right) \quad (10)$$

for all $n \geq n^*(\varepsilon)$.

For some given $c > 1$ we can always pick ε in $(0, 1)$ so that $(1-\varepsilon)c > 1$. With this selection we conclude from (9) that

$$\lim_{n \rightarrow \infty} P\left(n; \frac{(1-\varepsilon)c}{f_*} \rho_n^*\right) \leq \liminf_{n \rightarrow \infty} P(n; \rho_n)$$

hence $\liminf_{n \rightarrow \infty} P(n; \rho_n) = 1$ by the one-law at (8) (with c replaced by $(1-\varepsilon)c$). It is now plain that $\lim_{n \rightarrow \infty} P(n; \rho_n) = 1$ as desired.

Similar arguments apply *mutatis mutandis* to get the zero-law of Theorem 3.1: With $0 < c < 1$ we use (10) and the zero-law

at (8) (with c replaced by $(1 + \varepsilon)c$ where ε is selected so that $(1 + \varepsilon)c < 1$). Details are left to the interested reader. \square

Implications of Theorem 3.1 for power allocation are given in Section VIII-A, and pointers to earlier results are discussed in Sections VIII-B and VIII-C. It is worth noting that f_\star is the only artifact of the density function which enters Theorem 3.1—The actual location x_\star where the minimum is achieved plays no role as long as it is a point of continuity for f .

Theorem 3.1 identifies the scaling $\rho_F^\star : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ given by

$$\rho_{F,n}^\star = \frac{1}{f_\star} \cdot \frac{\log n}{n} = \frac{1}{f_\star} \cdot \rho_n^\star, \quad n = 1, 2, \dots \quad (11)$$

as a *critical* scaling for graph connectivity. Roughly speaking, for n large, a communication range ρ_n suitably larger (respectively, smaller) than $\rho_{F,n}^\star$ ensures that the graph $\mathbb{G}(n; \rho_n)$ is connected (respectively, disconnected) with very high probability if $\rho_n \sim c\rho_{F,n}^\star$ with $c > 1$ (respectively, $0 < c < 1$). It is customary [18, p. 376] to summarize (6)–(7) as a *strong* zero-one law, and to call the scaling $\rho_F^\star : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ a *strong* critical scaling. The boundary case $c = 1$ is more delicate and is partially handled with the help of the very strong zero-one law developed in [16]; see also [13] and [15] in the uniform case.

Theorem 3.1 also implies

$$\lim_{n \rightarrow \infty} P(n; \rho_n) = \begin{cases} 0 & \text{if } \lim_{n \rightarrow \infty} \frac{\rho_n}{\rho_{F,n}^\star} = 0 \\ 1 & \text{if } \lim_{n \rightarrow \infty} \frac{\rho_n}{\rho_{F,n}^\star} = \infty \end{cases} \quad (12)$$

with scaling $\rho : \mathbb{N}_0 \rightarrow \mathbb{R}_+$. According to (12), the one law (respectively, zero law) emerges when considering scalings $\rho : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ which are *at least* an order of magnitude larger (respectively, smaller) than ρ_F^\star . Contrast this with (6)–(7) where the one law (respectively, zero law) holds for scalings $\rho : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ which are larger (respectively, smaller) than ρ_F^\star but still of the *same* order of magnitude as ρ_F^\star ! It is therefore natural to refer to the situation (12) as a *weak* zero-one law and to call the scaling ρ_F^\star a *weak* critical scaling [18, p. 376].

Note that ρ^\star is also a weak critical scaling for connectivity under *any* distribution F satisfying the assumptions of Theorem 3.1, a somewhat robust, albeit weak, conclusion.

IV. AN EQUIVALENCE RESULT

Fix $n = 2, 3, \dots$. With the node locations X_1, \dots, X_n , we associate the rvs $X_{n,1}, \dots, X_{n,n}$ which are the locations of the n users arranged in increasing order, i.e., $X_{n,1} \leq \dots \leq X_{n,n}$ with the convention $X_{n,0} = 0$ and $X_{n,n+1} = 1$. The rvs $X_{n,1}, \dots, X_{n,n}$ are the *order statistics* associated with the rvs X_1, \dots, X_n ; they induce the *spacings*

$$L_{n,k} := X_{n,k} - X_{n,k-1}, \quad k = 1, \dots, n+1. \quad (13)$$

We also introduce the *maximal spacing* M_n as the rv defined by

$$M_n := \max(L_{n,k}, k = 2, \dots, n). \quad (14)$$

For each $\rho > 0$, the graph $\mathbb{G}(n; \rho)$ is connected if and only if $L_{n,k} \leq \rho$ for all $k = 2, \dots, n$, so that

$$P(n; \rho) = \mathbb{P}[M_n \leq \rho]. \quad (15)$$

The main technical contribution of this paper takes the following form.

Proposition 4.1: Under Assumptions 1 and 2 on F , we have the convergence

$$f_\star \frac{M_n}{\rho_n^\star} \xrightarrow{P} 1. \quad (16)$$

Proposition 4.1 is related to earlier results by Deheuvels [3, Theorem 4, p. 1183] and Penrose [20, Theorem 1.1, p. 247]; see the discussion in Sections VIII-B and VIII-C, respectively. The relevance of Proposition 4.1 to Theorem 3.1 lies in the following equivalence.

Lemma 4.2: Under the assumptions of Theorem 3.1, the convergence (7) under (6) is equivalent to (16)

Proof: We note that (16) is equivalent to

$$f_\star \frac{M_n}{\rho_n^\star} \xRightarrow{P} 1 \quad (17)$$

since the modes of convergence in distribution and in probability are equivalent when the limit is a constant [1, p. 25]. In particular, this amounts to

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[f_\star \frac{M_n}{\rho_n^\star} \leq c \right] = \begin{cases} 0 & \text{if } 0 < c < 1 \\ 1 & \text{if } 1 < c. \end{cases} \quad (18)$$

By virtue of (15) this last convergence is just a rewriting of (8), and the desired equivalence now follows from Corollary 3.2. \square

Thus, the zero-one law of Theorem 3.1 is an expression of the limiting property (16) exhibited by the sequence of maximal spacings $\{M_n, n = 2, \dots\}$. The proof of this convergence is developed in the next three sections.

V. BACK TO UNIFORM VARIATES

The first step consists in showing how the maximal spacings under F are determined by the order statistics under the uniform distribution.

To prepare the discussion, note that the mapping $F : [0, 1] \rightarrow [0, 1]$ is nondecreasing as a distribution function, hence admits a generalized inverse $F^\leftarrow : [0, 1] \rightarrow [0, 1]$ [23, Sec. 0.2]. However, under (3) the continuous mapping $F : [0, 1] \rightarrow [0, 1]$ is strictly increasing, hence invertible in the usual sense. Thus, the generalized inverse coincides with the usual inverse which is strictly increasing and continuous.

Under Assumption 1, the mapping $F : [0, 1] \rightarrow [0, 1]$ is absolutely continuous, hence differentiable a.e. on $[0, 1]$. From the obvious identity $F^\leftarrow(F(x)) = x$ on $[0, 1]$, we see that F^\leftarrow is differentiable at t if F is itself differentiable at $F^\leftarrow(t)$, in which case we have

$$\frac{d}{dt} F^\leftarrow(t) = g(t)^{-1}$$

with mapping $g : [0, 1] \rightarrow \mathbb{R}_+$ defined by

$$g(t) := f(F^\leftarrow(t)), \quad t \in [0, 1].$$

In fact, a little more can be said in that the inverse mapping $F^{\leftarrow} : [0, 1] \rightarrow [0, 1]$ is also absolutely continuous whenever $f_{\star} > 0$, and the relation

$$F^{\leftarrow}(t) = \int_0^t g(s)^{-1} ds, \quad t \in [0, 1] \quad (19)$$

therefore holds. Details are left to the interested reader. In addition to the i.i.d. $[0, 1]$ -valued rvs $\{X_i, i = 1, 2, \dots\}$, consider a second collection of i.i.d. rvs $\{U_i, i = 1, 2, \dots\}$ which are all uniformly distributed on $[0, 1]$ —For instance, we can take $U_i = F(X_i)$ for all $i = 1, 2, \dots$. In analogy with the earlier notation, for each $n = 2, 3, \dots$, we introduce the order statistics $U_{n,1}, \dots, U_{n,n}$ associated with the n i.i.d. rvs U_1, \dots, U_n , and we again use the convention $U_{n,0} = 0$ and $U_{n,n+1} = 1$. Key to our approach is the well-known stochastic equivalence [2, p. 15] that

$$(X_1, \dots, X_n) =_{st} (F^{\leftarrow}(U_1), \dots, F^{\leftarrow}(U_n))$$

so that

$$(X_{n,1}, \dots, X_{n,n}) =_{st} (F^{\leftarrow}(U_{n,1}), \dots, F^{\leftarrow}(U_{n,n})).$$

The representation

$$M_n =_{st} \max \left(\int_{U_{n,k-1}}^{U_{n,k}} g(s)^{-1} ds, \quad k = 2, \dots, n \right) \quad (20)$$

follows from (19) upon noting that

$$F^{\leftarrow}(U_{n,k}) - F^{\leftarrow}(U_{n,k-1}) = \int_{U_{n,k-1}}^{U_{n,k}} g(s)^{-1} ds$$

for each $k = 1, \dots, n+1$.

These observations suggest that the convergence (16) is likely to emerge through the asymptotic properties of the rvs $U_{n,1}, \dots, U_{n,n+1}$ modulated by the function f (via g).

VI. A USEFUL REPRESENTATION AND RELATED FACTS

In a second step we leverage the representation (20) by relying on the following representation of the order statistics $U_{n,1}, \dots, U_{n,n+1}$: Consider a collection $\{\xi_j, j = 1, 2, \dots\}$ of i.i.d. rvs which are exponentially distributed with unit parameter, and set

$$T_0 = 0, \quad T_k = \xi_1 + \dots + \xi_k, \quad k = 1, 2, \dots$$

For all $n = 1, 2, \dots$, the stochastic equivalence

$$(U_{n,1}, \dots, U_{n,n}) =_{st} \left(\frac{T_1}{T_{n+1}}, \dots, \frac{T_n}{T_{n+1}} \right) \quad (21)$$

is known to hold [22, p. 403] (and references therein).

In the remainder of this section, we explore some easy facts concerning maxima of i.i.d. exponentially distributed rvs. As the reader may have already guessed, these quantities (via (20) and (21)) will play a crucial role in the proof of Proposition 4.1. Thus, for each $n = 1, \dots$, let K_n denote a nonempty subset of $\{1, \dots, n+1\}$, and write $|K_n|$ for its cardinality. Also set

$$M(K_n) := \max(\xi_k, k \in K_n). \quad (22)$$

Lemma 6.1: The convergence

$$\frac{M(K_n)}{\log n} \xrightarrow{P} 1 \quad (23)$$

takes place whenever there exists some θ in $(0, 1]$ such that

$$\lim_{n \rightarrow \infty} \frac{|K_n|}{n} = \theta. \quad (24)$$

Proof: Fix $n = 3, 4, \dots$ and $t \geq 0$. By independence, we get

$$\begin{aligned} \mathbb{P}[M(K_n) \leq t] &= \mathbb{P}[\xi_k \leq t, k \in K_n] \\ &= (1 - e^{-t})^{|K_n|} \end{aligned}$$

so that

$$\begin{aligned} \mathbb{P}\left[\frac{M(K_n)}{\log n} \leq t\right] &= (1 - e^{-t \log n})^{|K_n|} \\ &= \left(1 - \frac{n^{1-t}}{n}\right)^{|K_n|} = \left(1 - \frac{n^{1-t}}{n}\right)^{\frac{|K_n|}{n}}. \end{aligned}$$

It is clear that

$$\lim_{n \rightarrow \infty} \left(1 - \frac{n^{1-t}}{n}\right)^n = \begin{cases} 0 & \text{if } 0 \leq t < 1 \\ 1 & \text{if } 1 < t \end{cases}$$

and with the help of (24) it is now straightforward to see that

$$\lim_{n \rightarrow \infty} \mathbb{P}\left[\frac{M(K_n)}{\log n} \leq t\right] = \begin{cases} 0 & \text{if } 0 \leq t < 1 \\ 1 & \text{if } 1 < t. \end{cases}$$

As this last convergence implies

$$\frac{M(K_n)}{\log n} \Rightarrow_n 1,$$

the convergence (23) follows from the fact that convergence in distribution is equivalent to convergence in probability when the limit is a constant ([1, p. 25]. \square)

Lemma 6.1 has a number of useful consequences which we now discuss. For each $n = 2, 3, \dots$, write

$$M_n^u := \max(L_{n,k}^u, k = 2, \dots, n) \quad (25)$$

with

$$L_{n,k}^u := U_{n,k} - U_{n,k-1}, \quad k = 1, \dots, n+1. \quad (26)$$

These quantities coincide with similar quantities given by (13) and (14), respectively, when F is taken to be the uniform distribution on $[0, 1]$. The following result is a byproduct of Lemma 6.1.

Lemma 6.2: Under the assumptions of Lemma 6.1 we also have

$$\frac{1}{\rho_n^*} \max(L_{n,k}^u, k \in K_n) \xrightarrow{P} 1. \quad (27)$$

Proof: By virtue of (26) and of the stochastic identity (21), we note that

$$(L_{n,1}^u, \dots, L_{n,n}^u) =_{st} \left(\frac{\xi_1}{T_{n+1}}, \dots, \frac{\xi_n}{T_{n+1}} \right).$$

Hence, in order to establish (27) we need only show that

$$\frac{1}{\rho_n^*} \max \left(\frac{\xi_k}{T_{n+1}}, k \in K_n \right) \xrightarrow{P} 1, \quad (28)$$

a convergence statement equivalent to

$$\frac{n}{T_{n+1}} \frac{M(K_n)}{\log n} \xrightarrow{P} 1. \quad (29)$$

The validity of (29) follows from Lemma 6.1 since

$$\lim_{n \rightarrow \infty} \frac{T_{n+1}}{n} = 1 \quad \text{a.s.} \quad (30)$$

by the Strong Law of Large Numbers. \square

Specializing Lemma 6.2 to $K_n = \{2, \dots, n\}$ (so that $\theta = 1$), we find

$$\frac{M_n^u}{\rho_n^*} \xrightarrow{P} 1. \quad (31)$$

This result was already obtained by Lévy [5], [17], and yields Theorem 3.1 (via Lemma 4.2) when F is the uniform distribution since then $f_\star = 1$. Slud has shown [26, Theorem 2.1, p. 343] that

$$nM_n^u - \log n = O(\log \log n) \quad \text{a.s.}$$

so that the convergence (31) does in fact hold in the stronger a.s. sense.

VII. A PROOF OF PROPOSITION 4.1

Fix $n = 2, 3, \dots$ Upon setting

$$Z_{n,k} := \int_{\frac{T_{k-1}}{T_{n+1}}}^{\frac{T_k}{T_{n+1}}} g(s)^{-1} ds, \quad k = 1, \dots, n+1,$$

we define the rv \widehat{M}_n given by

$$\widehat{M}_n := \max(Z_{n,k}, k = 2, \dots, n). \quad (32)$$

It is plain from (20) and (21) that $M_n =_{st} \widehat{M}_n$, and the convergence (16) will be established if we show that

$$f_\star \frac{\widehat{M}_n}{\rho_n^*} \xrightarrow{P} 1. \quad (33)$$

Thus, for every $\eta > 0$ we need to show that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\left| f_\star \frac{\widehat{M}_n}{\rho_n^*} - 1 \right| \geq \eta \right] = 0. \quad (34)$$

This is equivalent to the simultaneous validity of the two convergence statements

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[1 + \eta \leq f_\star \frac{\widehat{M}_n}{\rho_n^*} \right] = 0 \quad (35)$$

and

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[f_\star \frac{\widehat{M}_n}{\rho_n^*} \leq 1 - \eta \right] = 0. \quad (36)$$

A. Establishing the Convergence (35)

For each $n = 2, 3, \dots$, the easy upper bounds

$$Z_{n,k} \leq \frac{1}{f_\star} \cdot \frac{\xi_k}{T_{n+1}}, \quad k = 2, \dots, n$$

follow with the help of (2) from the inequalities

$$f_\star = f(x_\star) \leq g(t), \quad t \in [0, 1]. \quad (37)$$

This readily implies

$$f_\star \frac{\widehat{M}_n}{\rho_n^*} \leq \frac{n}{T_{n+1}} \cdot \frac{M(\{2, \dots, n\})}{\log n} \quad (38)$$

with $M(\{2, \dots, n\})$ given by (22) (with $K_n = \{2, \dots, n\}$). As in the proof of Lemma 6.2 (essentially (29)) we conclude that

$$\frac{n}{T_{n+1}} \cdot \frac{M(\{2, \dots, n\})}{\log n} \xrightarrow{P} 1. \quad (39)$$

Consequently, as the upper bound (38) implies

$$\mathbb{P} \left[1 + \eta \leq f_\star \frac{\widehat{M}_n}{\rho_n^*} \right] \leq \mathbb{P} \left[1 + \eta \leq \frac{n}{T_{n+1}} \cdot \frac{M(\{2, \dots, n\})}{\log n} \right]$$

for all $n = 2, 3, \dots$, we obtain the desired convergence (35) from (39) upon letting n go to infinity in this last inequality.

B. A Localization Argument

The proof of (36) is more involved and relies on a suitable lower bound for the maximum \widehat{M}_n . This bound is constructed by considering a subset of the rvs $Z_{n,2}, \dots, Z_{n,n}$ entering the definition (32). The basic idea amounts to the following *localization* argument: Pick any element x_\star in $[0, 1]$ which achieves the minimum of f as stated in Assumption 2. The distribution function F being continuous and strictly increasing, the value $t_\star := F(x_\star)$ is the unique element in $[0, 1]$ such that $F^\leftarrow(t_\star) = x_\star$. We then construct the lower bound by keeping only those values of $k = 2, \dots, n$ for which the endpoints of the interval $(\frac{T_{k-1}}{T_{n+1}}, \frac{T_k}{T_{n+1}})$ have a very high likelihood of being very close to t_\star as n grows large. In the limiting regime the values taken by g on such an interval $(\frac{T_{k-1}}{T_{n+1}}, \frac{T_k}{T_{n+1}})$ can be made arbitrarily close to $g(t_\star) = f_\star$, say no greater than $f_\star + \varepsilon$ for arbitrarily small $\varepsilon > 0$. A detailed construction is presented next.

For every $\varepsilon > 0$ we note from (4) that $|g(t) - g(t_\star)| \leq \varepsilon$ whenever t is selected in $[0, 1]$ such that $|F^\leftarrow(t) - F^\leftarrow(t_\star)| \leq \delta(\varepsilon)$ (with $\delta(\varepsilon) > 0$ ensuring (4)). Since (19) and (37) together imply

$$|F^\leftarrow(t) - F^\leftarrow(t_\star)| \leq \frac{1}{f_\star} \cdot |t - t_\star|, \quad t \in [0, 1]$$

it follows that

$$|g(t) - g(t_\star)| \leq \varepsilon \quad \text{if} \quad |t - t_\star| \leq \delta f_\star \quad (40)$$

with $0 < \delta \leq \delta(\varepsilon)$.

Under the enforced assumptions, we have $0 < x_* < 1$ (respectively, $x_* = 0, x_* = 1$) if and only if $0 < t_* < 1$ (respectively, $t_* = 0, t_* = 1$). Below we give a complete discussion for the case $0 < x_* < 1$, as the two other cases can be handled *mutatis mutandis*.

Thus, assume $0 < x_* < 1$ so that $0 < t_* < 1$, or equivalently,

$$0 < \min(t_*, 1 - t_*). \quad (41)$$

It is always possible to pick $\theta > 0$ such that

$$0 < \theta < \min(t_*, 1 - t_*) \quad (42)$$

in which case $t_* - \theta > 0$ and $t_* + \theta < 1$. For each $n = 2, 3, \dots$, we introduce the subset $K_n(\theta)$ of $\{1, \dots, n+1\}$ defined by

$$K_n(\theta) := \{\lceil n(t_* - \theta) \rceil, \dots, \lceil n(t_* + \theta) \rceil\}.$$

As we are interested in limiting results, we need only consider $n \geq n^*(\theta)$ with $n^*(\theta) = 2(t_* - \theta)^{-1}$ (as we do from now on), in which case $\lceil n(t_* - \theta) \rceil \geq 2$ and $K_n(\theta) \subseteq \{2, \dots, n\}$.

Fix $n = 2, 3, \dots$ with $n \geq n^*(\theta)$. It is plain that

$$\widehat{M}_n(\theta) \leq \widehat{M}_n \quad (43)$$

where we have set

$$\widehat{M}_n(\theta) := \max(Z_{n,k}, k \in K_n(\theta)).$$

To proceed, we observe the following elementary facts. For each $a = \pm 1$ and $b = 0, -1$, we have

$$\lim_{n \rightarrow \infty} \frac{\lceil n(t_* + a\theta) \rceil + b}{n} = t_* + a\theta$$

so that

$$\lim_{n \rightarrow \infty} \frac{T_{\lceil n(t_* + a\theta) \rceil + b}}{T_{n+1}} = t_* + a\theta \quad \text{a.s.} \quad (44)$$

by the Strong Law of Large Numbers. Building on this observation, given $\zeta > 0$, for each $n \geq n^*(\theta)$ we introduce the events

$$\Omega_n^{a,b}(\theta; \zeta) := \left[\left| \frac{T_{\lceil n(t_* + a\theta) \rceil + b}}{T_{n+1}} - (t_* + a\theta) \right| \leq \zeta \right]$$

for $a = \pm 1$ and $b = 0, -1$, and set

$$\Omega_n(\theta; \zeta) := \cap_{a=\pm 1, b=0, -1} \Omega_n^{a,b}(\theta; \zeta).$$

The convergence (44) then yields

$$\lim_{n \rightarrow \infty} \mathbb{P}[\Omega_n(\theta; \zeta)] = 1, \quad \zeta > 0. \quad (45)$$

Fix $n \geq n^*(\theta)$ and pick $\zeta > 0$ such that $\theta + \zeta < \min(t_*, 1 - t_*)$; such a choice for ζ is always possible under (42). On the event $\Omega_n(\theta; \zeta)$, it is automatically the case that

$$\left| \frac{T_{\lceil n(t_* - \theta) \rceil - 1}}{T_{n+1}} - (t_* - \theta) \right| \leq \zeta$$

and

$$\left| \frac{T_{\lceil n(t_* + \theta) \rceil}}{T_{n+1}} - (t_* + \theta) \right| \leq \zeta.$$

The inclusion

$$I_n(t_*; \theta) \subseteq [t_* - (\theta + \zeta), t_* + (\theta + \zeta)] \quad (46)$$

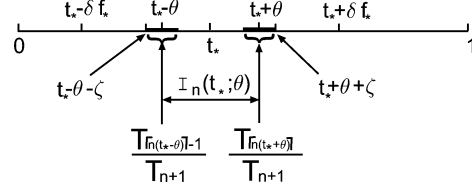


Fig. 1. The random interval $I_n(t_*, \theta)$.

now follows where $I_n(t_*; \theta)$ is the random interval given by

$$I_n(t_*; \theta) := \bigcup_{k \in K_n(\theta)} \left[\frac{T_{k-1}}{T_{n+1}}, \frac{T_k}{T_{n+1}} \right].$$

C. Establishing the Convergence (36)

Fix $\varepsilon > 0$ and δ in $(0, \delta(\varepsilon))$ where $\delta(\varepsilon)$ ensures (4). Pick θ in $(0, 1)$ and $\zeta > 0$ such that the conditions

$$\theta + \zeta < \delta f_* < \min(t_*, 1 - t_*)$$

hold—This is always possible under (41), possibly by reducing δ appropriately without affecting (40). With these choices, still on the event $\Omega_n(\theta; \zeta)$, we observe from (46) that the inclusions

$$I_n(t_*; \theta) \subseteq (t_* - \delta f_*, t_* + \delta f_*) \subseteq (0, 1)$$

hold. See Fig. 1. The two solid regions identify the ranges of possible values for the boundary points of the random interval $I_n(t_*; \theta)$, namely $\frac{T_{\lceil n(t_* - \theta) \rceil - 1}}{T_{n+1}}$ and $\frac{T_{\lceil n(t_* + \theta) \rceil}}{T_{n+1}}$.

From (40), we conclude that the inequalities

$$|g(t) - g(t_*)| \leq \varepsilon, \quad t \in I_n(t_*; \theta)$$

all hold, hence

$$f_* \leq g(t) \leq f_* + \varepsilon, \quad t \in I_n(t_*; \theta)$$

since $g(t_*) = f(x_*) = f_*$. In the notation (22) (with $K_n = K_n(\theta)$), the inequality

$$(f_* + \varepsilon)^{-1} \cdot \frac{M(K_n(\theta))}{T_{n+1}} \leq \widehat{M}_n(\theta) \quad (47)$$

readily follows, hence

$$(f_* + \varepsilon)^{-1} \cdot \frac{M(K_n(\theta))}{T_{n+1}} \leq \widehat{M}_n \quad (48)$$

by virtue of (43). Thus, on the event $\Omega_n(\theta; \zeta)$, for a given $\eta > 0$, the inequality $f_* \frac{\widehat{M}_n}{\rho_n^*} \leq 1 - \eta$ readily implies (via (48)) that

$$\frac{M(K_n(\theta))}{\log n} \cdot \frac{n}{T_{n+1}} \leq a(\eta; \varepsilon) \quad (49)$$

with

$$a(\eta; \varepsilon) := (1 - \eta) \cdot \frac{f_* + \varepsilon}{f_*}.$$

By standard bounding and decomposition arguments, we then get

$$\begin{aligned} & \mathbb{P} \left[f_{\star} \frac{\widehat{M}_n}{\rho_n^{\star}} \leq 1 - \eta \right] \\ & \leq \mathbb{P} \left[\left[\frac{M(K_n(\theta))}{\log n} \cdot \frac{n}{T_{n+1}} \leq a(\eta; \varepsilon) \right] \cap \Omega_n(\theta; \zeta) \right] \\ & \quad + \mathbb{P}[\Omega_n(\theta; \zeta)^c] \\ & = 1 - \mathbb{P}[\Omega_n(\theta; \zeta)] \\ & \quad + \mathbb{P} \left[\frac{M(K_n(\theta))}{\log n} \cdot \frac{n}{T_{n+1}} \leq a(\eta; \varepsilon) \right]. \end{aligned} \quad (50)$$

Note that (36) needs to be established only for $0 < \eta < 1$ for otherwise the convergence is trivially true. Thus, pick $0 < \eta < 1$ and note that $\varepsilon > 0$ can be selected sufficiently small such that $a(\eta; \varepsilon) < 1$ since this last condition is equivalent to

$$\varepsilon < \frac{\eta}{1 - \eta} \cdot f_{\star}.$$

With such a selection of $\varepsilon > 0$, we have

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\frac{M(K_n(\theta))}{\log n} \cdot \frac{n}{T_{n+1}} \leq a(\eta; \varepsilon) \right] = 0 \quad (51)$$

since

$$\frac{M(K_n(\theta))}{\log n} \cdot \frac{n}{T_{n+1}} \xrightarrow{P} 1.$$

This last convergence follows by combining (30) and Lemma 6.1 (with $K_n = K_n(\theta)$).² Finally, let n go to infinity in (50). The desired result (36) follows from (45) and (51).

The cases $x_{\star} = 0$ and $x_{\star} = 1$ can be analyzed in a similar way. Still with $t_{\star} = F(x_{\star})$, we have $t_{\star} = 0$ and $t_{\star} = 1$, respectively. As a result we need only change the definition of $K_n(\theta)$ to $\{2, \dots, \lceil n\theta \rceil\}$ and $\{\lceil n(1 - \theta) \rceil, \dots, n\}$, respectively, for n large enough in order to ensure $K_n(\theta) \subseteq \{2, \dots, n\}$. Details are left to the interested reader.

VIII. CONCLUDING REMARKS

A. Zero-One Laws and Critical Transmission Ranges

The one-dimensional model considered here arises in the same manner as the two-dimensional disk model by assuming a simplified pathloss, no user interference and no fading: Users (or interchangeably, nodes) all transmit at the same power level P . For distinct users located at X_i and X_j , say, their received power $P_{i,j}$ is assumed given by

$$P_{i,j} := P \cdot |X_i - X_j|^{-\nu}$$

for some pathloss exponent $\nu > 0$. Nodes i and j are then said to communicate if $P_{i,j} \geq \Gamma$ for some threshold $\Gamma > 0$. This condition is equivalent to requiring

$$|X_i - X_j| \leq \rho \quad \text{with} \quad \rho := \left(\frac{P}{\Gamma} \right)^{1/\nu}$$

²This is in essence the proof of Lemma 6.2; see (29).

and shows that the transmission range ρ is a proxy for the transmit power P .

A natural question consists in determining the minimum power level needed to ensure network connectivity amongst the nodes located at X_1, \dots, X_n . Expressed in terms of communication range, this amounts to considering the *critical transmission range* R_n defined as

$$R_n := \min(\rho > 0 : \mathbb{G}(n; \rho) \text{ is connected}).$$

However, being a function of X_1, \dots, X_n , the rv R_n has limited operational use since the node locations are neither available nor should their knowledge be expected, especially in the presence of mobility. Enters Proposition 4.1: The identity $R_n = M_n$ allows the critical scaling $\rho_F^{\star} : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ to be interpreted as a *deterministic* estimate of the critical transmission range in many node networks since $R_n \simeq \rho_{F,n}^{\star}$ with high probability for n large (as formalized by (16)). The corresponding critical power level is now given by

$$P_{F,n}^{\star} := \Gamma (\rho_{F,n}^{\star})^{\nu}, \quad n = 1, 2, \dots$$

Put differently, the network with n nodes transmitting at power level $cP_{F,n}^{\star}$ is connected (respectively, disconnected) with very high probability if $c > 1$ (respectively, $0 < c < 1$) for n large.

B. Connections With Earlier Results

Of particular interest are earlier results given by Deheuvels [3] under the following conditions: i) the density function f is continuous on $(0, 1)$; ii) the minimizer x_{\star} appearing in (3) is an *isolated* minimizer; iii) for some finite constant $r > 0$, we have $0 < d_r \leq D_r < \infty$ where³

$$\begin{aligned} d_r &:= \liminf_{h \rightarrow 0} \left(\frac{f(x_{\star} + h) - f(x_{\star})}{|h|^r} \right) \\ &\quad \text{and} \\ D_r &:= \limsup_{h \rightarrow 0} \left(\frac{f(x_{\star} + h) - f(x_{\star})}{|h|^r} \right). \end{aligned}$$

Under these conditions, Deheuvels [3, Theorem 4, p. 1183] (with $k = 1$) has shown that

$$-\frac{1}{r} = \liminf_{n \rightarrow \infty} \left(\frac{nf_{\star}M_n - \log n}{\log \log n} \right) \quad \text{a.s.} \quad (52)$$

and

$$\limsup_{n \rightarrow \infty} \left(\frac{nf_{\star}M_n - \log n}{\log \log n} \right) = 2 - \frac{1}{r} \quad \text{a.s.} \quad (53)$$

Therefore, noting that

$$\frac{nf_{\star}M_n}{\log n} - 1 = \frac{nf_{\star}M_n - \log n}{\log \log n} \cdot \frac{\log \log n}{\log n}$$

for all $n = 2, 3, \dots$, we readily see from (52) and (53) that the convergence (16) holds (in fact in the a.s. sense).

³This is the form that the conditions take when x_{\star} is an interior point of the interval $[0, 1]$. Obvious modifications need to be made when either $x_{\star} = 0$ or $x_{\star} = 1$.

Thus, an a.s. version of Proposition 4.1 is an easy byproduct of the results by Deheuvels [3] provided we assume conditions much stronger than the ones enforced in the present paper. In the work reported here, the conditions i)–iii) are not needed, but the convergence result (16) is established only in probability. As a result of this tradeoff we are able to give a simpler and more direct proof.

C. In Higher Dimensions

The convergence (16) is compatible with a multidimensional result obtained by Penrose [20]: Theorem 1.1 of [20, p. 247] was discussed under the dimensional assumption $d \geq 2$ by methods very different from the ones used here. Yet, formally setting $d = 1$ in it, we recover (16) but in the a.s. sense.

D. The Case of Vanishing Densities

When $f_\star = 0$, a blind application of (11) yields $\rho_{F,n}^\star = \infty$ for all $n = 1, 2, \dots$. This begs the question as to what becomes of Theorem 3.1. Direct inspection shows that (16) cannot hold when $f_\star = 0$, thereby precluding the existence of a strong zero-one law (by the equivalence of Lemma 4.2). In fact, with a node placement distribution of the form

$$F(x) = x^{p+1}, \quad x \in [0, 1]$$

for some $p > 0$, the authors have shown [14] that only a weak zero-one law holds with (weak) critical scaling given by

$$\rho_{F,n}^\star = n^{-\frac{1}{p+1}}, \quad n = 1, 2, \dots$$

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Guang Han received the B.Eng. and M.Eng. degrees from Tsinghua University, China, and the Ph.D. degree from the University of Maryland, College Park, in 2000, 2002, and 2007, respectively, all in electrical engineering.

Since 2007, he has been a Senior Staff Electrical Engineer with Networks Advanced Technologies at Motorola, Arlington Heights, IL. His research interests include wireless *ad hoc* networks, wireless broadband networks, and random graph theory.

Armand M. Makowski (M'83–SM'94–F'06) received the Licence en Sciences Mathématiques degree from the Université Libre de Bruxelles, Belgium, in 1975, the M.S. degree in engineering-systems science from University of California, Los Angeles, in 1976, and the Ph.D. degree in applied mathematics from the University of Kentucky, Lexington, in 1981.

In August 1981, he joined the faculty of the Electrical Engineering Department at the University of Maryland, College Park, where he is Professor of Electrical and Computer Engineering. He has held a joint appointment with the Institute for Systems Research since its establishment in 1985. His research interests lie in applying advanced methods from the theory of stochastic processes to the modeling, design, and performance evaluation of engineering systems, with particular emphasis on communication systems and networks.

Dr. Makowski was a C.R.B. Fellow of the Belgian–American Educational Foundation (BAEF) for the academic year 1975–1976. He is also a 1984 recipient of the NSF Presidential Young Investigator Award.